Matricial Algebra and Numerical Analysis MANA

CM 2 : Bilinear Algebra and analytic geometry

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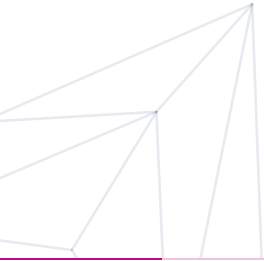
Motivation

Previously in MANA:

- review on linear algebra,
- introduction to isometries.



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When transformations that are not isometries are applied, it can change:

- lengths,
- angles,
- surfaces and volumes...

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Before we introduce transformations affecting these measures, it is necessary to introduce tools to calculate and manipulate them.

This is the purpose of this lesson.

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01

Norm, length and distance



Norms and length

Definition

Let V be a vector space. Is called norm the function $\|\cdot\|V \to \mathbb{R}$ which associate to $\mathbf{x} \in V$ the length $\|\mathbf{x}\|$, such that $\forall \lambda \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in V$ the following conditions are satisfied:

- Absolutely homogeneous: $\|\lambda \mathbf{x}\| = |\lambda| \|\mathbf{x}\|$,
- Triangle inequality: $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$,
- Positive definite: $\|\mathbf{x}\| \geq 0$ and $\|\mathbf{x}\| = 0 \Leftrightarrow \mathbf{x} = \mathbf{0}_V$.





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Distance

Given two vectors $\mathbf{x}, \mathbf{y} \in V$ and a norm $\|\cdot\| : V \to \mathbb{R}$, is called a distance or a metric between $\mathbf{x}$ and $\mathbf{y}$ the quantity:

$$d(\mathbf{x}, \mathbf{y}) := \|\mathbf{x} - \mathbf{y}\|. \tag{1}$$

The Manhattan distance, the Euclidian distance and the Tchebytchev distance stem from the norms seen previously.

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Distances and metric space

Distance

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Metric space

Let V be a vector space. The couple (V, d) is a metric space if the application $d : V \times V \to \mathbb{R}$ satisfies the three following conditions:

- Positive definite: $d(\mathbf{x}, \mathbf{y}) \geq 0 \, \forall \mathbf{x}, \mathbf{y} \in V$ and $d(\mathbf{x}, \mathbf{y}) = 0 \Leftrightarrow \mathbf{x} = \mathbf{y}$,
- Symmetry: $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$,
- Triangle inequality: $d(\mathbf{x}, \mathbf{z}) \leq d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}) \, \forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in V$.

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02

Inner product, angles and projection

Inner product

Definition

Let V be a vector space of $E = \mathbb{R}$ or $\mathbb{C}$. Is called an inner product the application $\langle \cdot, \cdot \rangle : V \times V \to E$ such that $\forall \lambda, \mu \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ the following conditions are satisfied:

- Bilinearity: $\langle \lambda \mathbf{x} + \mu \mathbf{y}, \mathbf{z} \rangle = \lambda \langle \mathbf{x}, \mathbf{z} \rangle + \mu \langle \mathbf{y}, \mathbf{z} \rangle$ and $\langle \mathbf{x}, \lambda \mathbf{y} + \mu \mathbf{z} \rangle = \lambda \langle \mathbf{x}, \mathbf{y} \rangle + \mu \langle \mathbf{x}, \mathbf{z} \rangle$,
- Symmetry: $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$,
- Positive definite: $\langle \mathbf{x}, \mathbf{x} \rangle > 0$ and $\langle \mathbf{x}, \mathbf{x} \rangle = 0 \Leftrightarrow \mathbf{x} = \mathbf{0}_V$.

Exercice: Check that the confitions are verified for the dot product.

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Inner product and symmetric positive definite matrices

Let the Euclidiant basis $V \in \mathbb{R}^N$ written $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_N\}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$ then the dot product can be written as:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \langle \sum_{i=1}^{N} x_i \mathbf{e}_i, \sum_{j=1}^{N} y_j \mathbf{e}_j \rangle \underset{\text{bilinearity}}{=} \sum_{i=1}^{n} \sum_{j=1}^{n} x_i \langle \mathbf{e}_i, \mathbf{e}_j \rangle y_j = \mathbf{x}^t \mathbf{I} \mathbf{d} \mathbf{y} = \mathbf{x}^t \mathbf{y}. \quad (2)$$

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Let some basis $B = \{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_N\}$ of V and $\mathbf{u}, \mathbf{v}$ the coordinate of $\mathbf{x}, \mathbf{y}$ in this basis, then we can write:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \langle \sum_{i=1}^{N} u_i \mathbf{b}_i, \sum_{j=1}^{N} v_j \mathbf{b}_j \rangle \underset{\text{bilinearity}}{=} \sum_{i=1}^{n} \sum_{j=1}^{n} u_i \langle \mathbf{b}_i, \mathbf{b}_j \rangle v_j = \mathbf{u}^t \mathbf{B} \mathbf{v}, \tag{3}$$

where $\mathbf{B}$ is, by definition of the inner product, a symmetric positive definite matrix. Similarly, if $\mathbf{B}$ is a symmetric positive definite matrix, then it can describe an inner product.

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Other definition and property

Norms induced by inner products

A norm can be induced by an inner product (this is not always the case). In such a case, it is defined as the following:

$$\|\mathbf{x}\| := \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} \tag{4}$$

This is the case of the ℓ_2 norm.

Property

Let $\mathbf{x} \in U$ a vector subspace of $\mathbb{R}^M$, $\mathbf{y} \in V$ a vector subspace of $\mathbb{R}^N$, $\mathbf{M} \in \mathcal{M}_{M,N}(\mathbb{R})$ and $\langle \cdot, \cdot \rangle$ an inner product, then:

$$\langle \mathbf{x}, \mathbf{M}\mathbf{y} \rangle = \langle \mathbf{M}^t \mathbf{x}, \mathbf{y} \rangle. \tag{5}$$

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Angle between two vectors

Let $\mathbf{x}, \mathbf{y} \in V$, $\mathbf{x}, \mathbf{y} \neq \mathbf{0}_V$, then the angle ϕ between the two vectors is such that:

$$\cos(\phi) = \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\| \|\mathbf{y}\|}. \tag{6}$$



We find again that if $\mathbf{x}$ and $\mathbf{y}$ are orthogonal then $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ and if they are collinear then the inner product is maximal.

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Orhtogonal projection on a subspace

Consider $\mathbf{x} \in V$ and a vector subspace $U \subset V$, described by the vectors $\mathbf{m}_1, \ldots, \mathbf{m}_N$. Considere $\mathbf{M} = \begin{pmatrix} \mathbf{m}_1 & \ldots & \mathbf{m}_N \end{pmatrix}$ where the vectors $\mathbf{m}_i$ are the columns of $\mathbf{M}$. Then the orthogonal projection of $\mathbf{x}$ on U is given by:

$$\Pi_U(\mathbf{x}) = \mathbf{M}(\mathbf{M}^t\mathbf{M})^{-1}\mathbf{M}^t\mathbf{x}. \tag{7}$$

The matrix

$$\mathbf{P}_U = \mathbf{M}(\mathbf{M}^t\mathbf{M})^{-1}\mathbf{M}^t \tag{8}$$

is called a projection matrix.

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Where does this formula comes from?

We are looking for the point $\Pi_U(\mathbf{x}) \in U$ the closest to $\mathbf{x}$.

- "$\Pi_U(\mathbf{x}) \in U$" $\Rightarrow \exists\, \boldsymbol{\lambda} \in \mathbb{R}^N$ such that $\Pi_U(\mathbf{x}) = \sum_{i=1}^{N} \boldsymbol{\lambda}_i \mathbf{m}_i = \mathbf{M}\boldsymbol{\lambda}$. It is similar to searche the coordinates $\boldsymbol{\lambda}$ of the point $\Pi_U(\mathbf{x})$ in the basis composed with the $\mathbf{m}_i$.
- "the closest to $\mathbf{x}$" that is such that the distance $\|\mathbf{x} - \Pi_U(\mathbf{x})\|^2$ is minimal. It implies that $\mathbf{x} - \Pi_U(\mathbf{x})$ must be orthogonal to all the $\mathbf{m}_i$.

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$$
\begin{aligned}
& \forall i = 1, \dots, N,\ \langle \mathbf{x} - \Pi_U(\mathbf{x}), \mathbf{m}_i \rangle = 0 \\
\Leftrightarrow\ & \forall i = 1, \dots, N,\ \mathbf{m}_i^t \left(\mathbf{x} - \mathbf{M}\boldsymbol{\lambda}\right) = 0 \\
\Leftrightarrow\ & \mathbf{M}^t(\mathbf{x} - \mathbf{M}\boldsymbol{\lambda}) = 0 \\
\Leftrightarrow\ & \mathbf{M}^t\mathbf{x} = \mathbf{M}^t\mathbf{M}\boldsymbol{\lambda} \\
\Leftrightarrow\ & \boldsymbol{\lambda} = (\mathbf{M}^t\mathbf{M})^{-1}\mathbf{M}^t\mathbf{x} \\
\Leftrightarrow\ & \Pi_U(\mathbf{x}) = \mathbf{M}(\mathbf{M}^t\mathbf{M})^{-1}\mathbf{M}^t\mathbf{x}
\end{aligned}
\tag{9}
$$

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Determinant

Consider $V = \{\mathbf{v}_1, \dots, \mathbf{v}_M\}$ a set of linearly inedendent vectors of $\mathbb{R}^N$, then the determinant of the matrix $\mathbf{M} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_M \end{pmatrix}$, noted $\det(\mathbf{M})$, gives the volume described by the vectors of V, that is the volume of the M-dimensional parallelepiped (parallelogram in 2D) where the sides are the vectors.

Remark : when transforming an object with the matrix M, the volume of the object is multiplied by $|\det(\mathbf{M})|$.

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Remark : when transforming an object with the matrix M, the volume of the object is multiplied by $|\det(\mathbf{M})|$.

What happens if $M \neq N$ or if $\det(\mathbf{M}) = 0$?

If $\det(\mathbf{M}) = 0$, then there is a linear dependence and thus the volume is 0. We then have to change of basis to restrict ourselves to the dimension equaled to the rank of $\mathbf{M}$.

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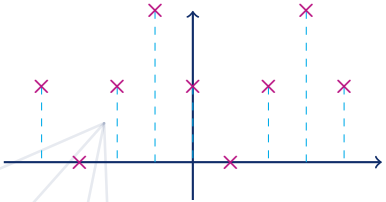
03

To the infinite dimension

Finite dimension VS infinite dimension space

Finite dimension

To the discrete index $i \in I \subset \mathbb{N}$ associate the value $v_i \in E \subset \mathbb{R}$ in $\mathbb{C}$.

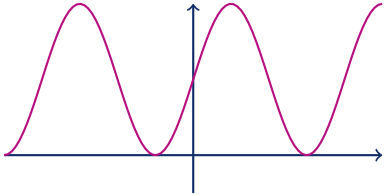


Operator of the finite sum:

$$\sum_{i \in I} v_i$$

Infinite dimension

To the continuous variable $x \in I \subset \mathbb{R}$ associate the value $v(x) \in E \subset \mathbb{R}$ or $\mathbb{C}$.



Operatorr of the infinite sum:

$$\int_I v(x)\mathrm{d}x$$

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Inner product of functions

Given two functions $u, v : \mathbb{R} \to \mathbb{R}$, then the inner product can be define as the following integral:

$$\langle u, v \rangle := \int_R u(x)v(x)dx \tag{10}$$

As in finite dimension from this inner product can be derived norms, etc...

To be continued in ITSI during the S6...

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Questions?

